BETHE ALGEBRA AND ALGEBRA OF FUNCTIONS ON THE SPACE OF DIFFERENTIAL OPERATORS OF ORDER TWO WITH POLYNOMIAL SOLUTIONS

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Abstract. We show that the following two algebras are isomorphic. The first is the algebra A_P of functions on the scheme of monic linear second-order differential operators on $\mathbb{C}$ with prescribed regular singular points at $z_1, \dots, z_n, \infty$, prescribed exponents $\Lambda^{(1)}, \dots, \Lambda^{(n)}, \Lambda^{(\infty)}$ at the singular points, and having the kernel consisting of polynomials only. The second is the Bethe algebra of commuting linear operators, acting on the vector space $\mathrm{Sing}\, L_{\Lambda^{(1)}} \otimes \dots \otimes L_{\Lambda^{(n)}}[\Lambda^{(\infty)}]$ of singular vectors of weight $\Lambda^{(\infty)}$ in the tensor product of finite dimensional polynomial $\mathfrak{gl}_2$-modules with highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$.

1. Introduction

1.1. There is a classical connection between Schubert calculus and representation theory of the Lie algebra $\mathfrak{gl}_N$. Let V be a vector space. Then Schubert cycles in the Grassmannian of N-dimensional subspaces of V are labeled by highest weights of polynomial irreducible $\mathfrak{gl}_N$-modules and if the intersection of several cycles is finite, then the intersection number is equal to the multiplicity of the unique one-dimensional representation in the tensor product of the corresponding polynomial finite-dimensional $\mathfrak{gl}_N$-modules. It is a challenge to understand in a deeper way this numerological relation, see [F], [B].

In this paper we prove a result which may help to comprehend better the interrelation of Schubert calculus and representation theory. Namely, for $N = 2$ under certain conditions, we identify the algebra of functions on the intersection of Schubert cycles with the Bethe algebra of linear operators acting on the multiplicity space of the one-dimensional subrepresentation.

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1.2. Let $\Lambda^{(1)}, \dots, \Lambda^{(n)}, \Lambda^{(\infty)}$ be dominant integral $\mathfrak{gl}_N$-weights. Consider the tensor product $L_{\boldsymbol{\Lambda}} = L_{\Lambda^{(1)}} \otimes \dots \otimes L_{\Lambda^{(n)}}$ of n polynomial irreducible finite-dimensional $\mathfrak{gl}_N$-modules with highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$, respectively. Let $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \subset L_{\boldsymbol{\Lambda}}$ be the subspace of singular vectors of weight $\Lambda^{(\infty)}$. Fix n distinct complex numbers $z_1, \dots, z_n$. Then the theory of the integrable Gaudin model provides us with a collection of commuting linear operators on that space, the operators being called the higher Gaudin Hamiltonians or the higher transfer matrices. The unital algebra A_L of endomorphisms of $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$, generated by the higher Gaudin Hamiltonians, is called the Bethe algebra.

Thus, given a set of $n+1$ highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}, \Lambda^{(\infty)}$ and a collection of complex numbers $z_1, \dots, z_n$ we construct the vector space $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and the commutative Bethe algebra of linear operators acting on that space.

There is another construction which starts with the same initial data. Having a set of highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}, \Lambda^{(\infty)}$ as above and a collection of distinct complex numbers $z_1, \dots, z_n$, we may construct one more vector space of the same dimension as $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and an algebra of commuting linear operators acting on that new space.

Namely, write $\Lambda^{(i)} = (\Lambda^{(i)}_1, \dots, \Lambda^{(i)}_N)$, $i = 1, \dots, n, \infty$, with $\Lambda^{(i)}_1 \geqslant \dots \geqslant \Lambda^{(i)}_{N-1} \geqslant \Lambda^{(i)}_N$ being non-negative integers. Consider the vector space $\mathbb{C}_d[x]$ of polynomials in x of degree not greater than d, where d is a natural number big enough with respect to n and N. Define $n+1$ Schubert cycles $C_{z_1,\Lambda^{(1)}}, \dots, C_{z_n,\Lambda^{(n)}}, C_{\infty,\Lambda^{(\infty)}}$ in the Grassmannian of all N-dimensional subspaces of $\mathbb{C}_d[x]$ as follows. For $i = 1, \dots, n$, the cycle $C_{z_i,\Lambda^{(i)}}$ is the closure of the set of all N-dimensional subspaces $V \subset \mathbb{C}_d[x]$ having a basis $f_1, \dots, f_N$ such that $f_j(x) = (x-z_i)^{\Lambda^{(i)}_j + N - j} + O((x-z_i)^{\Lambda^{(i)}_j + N - j + 1})$ for all j. The cycle $C_{\infty,\Lambda^{(\infty)}}$ is the closure of the set of all N-dimensional subspaces $V \subset \mathbb{C}_d[x]$ having a basis $f_1, \dots, f_N$ of polynomials of degrees $\Lambda^{(\infty)}_N, \Lambda^{(\infty)}_{N-1} + 1, \dots, \Lambda^{(i)}_{\infty} + N - 1$, respectively. Consider the intersection of these cycles and the algebra A_G of functions on this intersection.

By Schubert calculus, the dimension of A_G, regarded as a vector space, equals the dimension of the vector space $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Multiplication in the algebra A_G defines on the vector space A_G the commutative algebra of linear multiplication operators. The vector space A_G with the commutative algebra of multiplication operators is our new object.

We conjecture that there exists a natural isomorphism of the vector spaces $A_G \to \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ which induces an isomorphism of the corresponding algebras — the algebra of multiplication operators on A_G and the Bethe algebra A_L acting on $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.

Note that the Bethe algebra A_L has linear algebraic nature (it is generated by a finite set of relatively explicitly defined matrices) while the algebra A_G has geometric nature (it is the algebra of functions on the intersection of several algebraic cycles). An isomorphism of A_L and A_G may allow us to study one of the algebras in terms of the other.

For example, the intersection of Schubert cycles $C_{z_1,\Lambda^{(1)}},\dots,C_{z_n,\Lambda^{(n)}},C_{\infty,\Lambda^{(\infty)}}$ is not transversal if and only if the algebra A_G has nilpotent elements. Probably it is easier to check the presence of such elements in A_L than in A_G.

As another example, assume that all elements of the Bethe algebra A_L are diagonalizable. In that case the algebra A_G does not have nilpotent elements, hence the intersection of the Schubert cycles is transversal. Returning back to the Bethe algebra A_L we may conclude that the spectrum of A_L is simple.

The main result of this paper is the construction of an isomorphism of A_L and A_G for $N=2$.

1.3. The paper has the following structure.

In Section 2 we define two algebras A_M and A_D. The algebra A_M is the algebra generated by the Gaudin Hamiltonians acting of the subspace $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ of singular vectors of weight $\Lambda^{(\infty)}$ in the tensor product $M_{\boldsymbol{\Lambda}}=M_{\Lambda^{(1)}}\otimes\dots\otimes M_{\Lambda^{(n)}}$ of Verma $\mathfrak{gl}_2$-modules. Here $\Lambda^{(i)}=(m_s,0)$ for $i=1,\dots,n$ and $\Lambda^{(\infty)}=(\sum_{s=1}^n m_s-l,l)$.

To define the algebra A_D we consider the scheme C_D of monic linear second-order differential operators on $\mathbb{C}$ having regular singular points at $z_1,\dots,z_n,\infty$, with exponents $0,m_i+1$ at z_i for $i=1,\dots,n$, and exponents $-l,l-1-\sum_{s=1}^n m_s$ at infinity, and also having a polynomial of degree l in its kernel. Then we define A_D as the algebra of functions on C_D.

In Section 2.5 we construct an algebra epimorphism $\psi_{DM}:A_D\to A_M$.

In Section 3 we describe Sklyanin's separation of variables for the $\mathfrak{gl}_2$ Gaudin model and introduce the universal weight function. The important result of Section 3 is Theorem 3.4.2 on the Bethe ansatz method, which describes the interaction of the three objects: algebras A_M, A_D, and the universal weight function.

In Section 4 we consider the space A_D^*, dual to the vector space A_D, and the algebra of linear operators on A_D^* dual to the multiplication operators on A_D. Using the universal weight function we construct a linear map $\tau: A_D^*\to \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Theorem 4.3.1 says that τ is an isomorphism identifying the algebra of operators on A_D^* dual to multiplication operators and the Bethe algebra A_M acting on $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Theorem 4.3.1 is our first main result.

In Section 4.4 using the Grothendieck bilinear form on A_D we construct an isomorphism $\phi: A_D\to A_D^*$. The isomorphism ϕ identifies the algebra of multiplication operators on A_D with the algebra of operators on A_D^* dual to multiplication operators.

In Section 5 we introduce three more algebras A_G, A_P, A_L.

The algebra A_G is the algebra of functions on the intersection of Schubert cycles $C_{z_1,\Lambda^{(1)}},\dots,C_{z_1,\Lambda^{(n)}},C_{\infty,\Lambda^{(\infty)}}$ in the Grassmannian of two-dimensional subspaces of $\mathbb{C}_d[x]$.

To define the algebra A_P we consider the scheme C_P of monic linear second-order differential operators on $\mathbb{C}$ having regular singular points at $z_1,\dots,z_n,\infty$, with exponents $0,m_i+1$ at z_i for $i=1,\dots,n$ and exponents $-l,l-1-\sum_{s=1}^n m_s$ at infinity, and also

having the kernel consisting of polynomials only. Then the algebra A_P is the algebra of functions on C_P.

The algebra A_M is the algebra generated by the Gaudin Hamiltonians acting of the subspace $\operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$ of singular vectors of weight $\Lambda^{(\infty)}$ in the tensor product $L_{\boldsymbol\Lambda} = L_{\Lambda^{(1)}} \otimes \dots \otimes L_{\Lambda^{(n)}}$ of polynomial irreducible finite-dimensional $\mathfrak{gl}_N$-modules with highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$, respectively.

In Section 6 we discuss interrelations of the five algebras A_D, A_M, A_G, A_P, A_L. In particular, we have a natural isomorphism $\psi_{GP} : A_G \to A_P$.

In Section 6 we construct a linear map $\zeta : A_P \to \operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$. Using our first main result we show in Theorem 6.4.1 that ζ is an isomorphism identifying the algebra of multiplication operators on A_P and the Bethe algebra A_L acting on $\operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$. Theorem 6.4.1 is our second main result.

In Section 7 using the Shapovalov form on $\operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$ and the isomorphism ζ we construct a linear map $\theta : A_P^* \to \operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$. In Theorem 7.2.1 we show that θ is an isomorphism identifying the algebra on A_P^* of operators dual to multiplication operators and the Bethe algebra A_L acting on $\operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$. This is our third main result.

As an application of the third main result we prove the following statement, see Corollary 7.2.3.

If a two-dimensional vector space V belongs to the intersection of the Schubert cycles $C_{z_1,\Lambda^{(1)}}, \dots, C_{z_1,\Lambda^{(n)}}, C_{\infty,\Lambda^{(\infty)}}$ and if $d^2/dx^2 + a(x)d/dx + b(x)$ is the differential operator annihilating V, then there exists a nonzero eigenvector $v \in \operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$ of the Bethe algebra A_L with eigenvalues given by the functions $a(x)$ and $b(x)$.

Note that the converse statement follows from Corollaries 12.2.1 and 12.2.2 in [MTV3], see Sections 7.2.2 and 7.2.3.

In Appendix we discuss the relations between the Grothendieck residue on A_D, the Shapovalov form on $\operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$ and the homomorphism $A_D \to \operatorname{Sing} M_{\boldsymbol\Lambda}[\Lambda^{(\infty)}] \to \operatorname{Sing} L_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$.

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2. Two algebras

2.1. Algebra A_M.

2.1.1. Let $\mathfrak{gl}_2$ be the complex Lie algebra of 2×2-matrices with standard generators $e_{ab}, a, b = 1, 2$. Let $\mathfrak{h} \subset \mathfrak{gl}_2$ be the Cartan subalgebra of diagonal matrices, $\mathfrak{h}^*$ the dual space, (,) the standard scalar product on $\mathfrak{h}^*$, $\epsilon_1, \epsilon_2 \in \mathfrak{h}^*$ the standard orthonormal basis, $\alpha = \epsilon_1 - \epsilon_2$ the simple root.

Let $\boldsymbol\Lambda = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ be a collection of $\mathfrak{gl}_2$-weights, where $\Lambda^{(s)} = m_s\epsilon_1$ with $m_s \in \mathbb{C}$. Let l be a nonnegative integer. Define the $\mathfrak{gl}_2$-weight $\Lambda^{(\infty)} = \sum_{s=1}^n \Lambda^{(s)} - l\,\alpha$.

The pair $\boldsymbol\Lambda$, l is called *separating* if $\sum_{s=1}^n m_s - 2l + 1 + i \neq 0$ for all $i = 1, \dots, l$.

2.1.2. Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a collection of distinct complex numbers. Let

$$M_{\boldsymbol{\Lambda}} \;=\; M_{\Lambda^{(1)}} \otimes \dots \otimes M_{\Lambda^{(n)}}$$

be the tensor product of Verma $\mathfrak{gl}_2$-modules with highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$, respectively. Denote by $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ the subspace of $M_{\boldsymbol{\Lambda}}$ of singular vectors of weight $\Lambda^{(\infty)}$,

$$\mathrm{Sing}\ M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \;=\; \{\, v \in M_{\boldsymbol{\Lambda}} \mid e_{12}v = 0,\ e_{22}v = lv \,\}\ .$$

Consider the differential operator

$$\mathcal{D}_{M_{\boldsymbol{\Lambda}}} \;=\; \left(\frac{d}{dx} - \sum_{s=1}^{n} \frac{e_{11}^{(s)}}{x - z_s}\right)\left(\frac{d}{dx} - \sum_{s=1}^{n} \frac{e_{22}^{(s)}}{x - z_s}\right) - \left(\sum_{s=1}^{n} \frac{e_{21}^{(s)}}{x - z_s}\right)\left(\sum_{s=1}^{n} \frac{e_{12}^{(s)}}{x - z_s}\right).$$

The differential operator acts on $M_{\boldsymbol{\Lambda}}$-valued functions in x and is called *the universal differential operator* associated with $M_{\boldsymbol{\Lambda}}$ and $\boldsymbol{z}$, [T], [MTV1], [MTV3]. We have

$$\mathcal{D}_{M_{\boldsymbol{\Lambda}}} \;=\; \frac{d^2}{dx^2} \;-\; \sum_{s=1}^{n} \frac{m_s}{x - z_s}\frac{d}{dx} \;+\; \sum_{s=1}^{n} \frac{\widetilde{H}_s}{x - z_s} \tag{2.1}$$

where $\widetilde{H}_1, \dots, \widetilde{H}_n \in \mathrm{End}\,(M_{\boldsymbol{\Lambda}})$,

$$\widetilde{H}_s \;=\; \sum_{r \neq s} \frac{1}{z_s - z_r}\,(\, m_s m_r - \Omega_{s,r}\,) \qquad \text{and} \qquad \Omega_{s,r} \;=\; \sum_{i,j=1}^{2} e_{ij}^{(s)} \otimes e_{ji}^{(r)}\ . \tag{2.2}$$

We have $\widetilde{H}_1 + \dots + \widetilde{H}_n = 0$.

The operators $\widetilde{H}_1, \dots, \widetilde{H}_n$ are called *the Gaudin Hamiltonians* associated with $M_{\boldsymbol{\Lambda}}$ and $\boldsymbol{z}$. The Gaudin Hamiltonians have the following properties:

(i) The Gaudin Hamiltonians commute: $[\widetilde{H}_i, \widetilde{H}_j] = 0$ for all i, j.

(ii) The Gaudin Hamiltonians commute with the $\mathfrak{gl}_2$-action on $M_{\boldsymbol{\Lambda}}$: $[\widetilde{H}_i, x] = 0$ for all i and $x \in U(\mathfrak{gl}_2)$.

In particular, the Gaudin Hamiltonians preserve the subspace $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \subset M_{\boldsymbol{\Lambda}}$.

Restricting $\mathcal{D}_{M_{\boldsymbol{\Lambda}}}$ to the subspace of $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$-valued functions we obtain the differential operator

$$\mathcal{D}_{\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}} \;=\; \frac{d^2}{dx^2} \;-\; \sum_{s=1}^{n} \frac{m_s}{x - z_s}\frac{d}{dx} \;+\; \sum_{s=1}^{n} \frac{H_s}{x - z_s} \tag{2.3}$$

where $H_s = \widetilde{H}_s|_{\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]} \in \mathrm{End}\,(\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}])$.

The operator $\mathcal{D}_{\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}}$ will be called *the universal differential operator* associated with $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and $\boldsymbol{z}$. The operators $H_1, \dots, H_n$ will be called *the Gaudin Hamiltonians* associated with $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and $\boldsymbol{z}$.

The commutative unital subalgebra of $\mathrm{End}\,(\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}])$ generated by the Gaudin Hamiltonians $H_1, \dots, H_n$ will be called *the Bethe algebra* associated with $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and $\boldsymbol{z}$ and denoted by A_M.

2.1.3. Introduce the operators $G_0, \dots, G_{n-2}$ by the formula

$$\sum_{s=1}^{n} \frac{H_s}{x - z_s} \;=\; \frac{G_0 x^{n-2} + \dots + G_{n-2}}{(x-z_1)\dots(x-z_n)}\;.$$

Then $G_0 \,=\, l\,(\,\sum_{s=1}^{n} m_s \,+\, 1 \,-\, l\,)\,.$

2.1.4. **Lemma.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then*

$$\dim\, \mathrm{Sing}\; M_{\boldsymbol{\Lambda}}\big[\,\textstyle\sum_{s=1}^{n} \Lambda^{(s)} \,-\, l\,\alpha\big] \;=\; \dim\, M_{\boldsymbol{\Lambda}}\big[\,\textstyle\sum_{s=1}^{n} \Lambda^{(s)} \,-\, l\,\alpha\big] \;-\; \dim\, M_{\boldsymbol{\Lambda}}\big[\,\textstyle\sum_{s=1}^{n} \Lambda^{(s)} \,-\, (l-1)\,\alpha\big]\;.$$

Proof. The map $e_{12}e_{21} : M_{\boldsymbol{\Lambda}}\big[\,\sum_{s=1}^{n} \Lambda^{(s)} - (l-1)\,\alpha\big] \to M_{\boldsymbol{\Lambda}}\big[\,\sum_{s=1}^{n} \Lambda^{(s)} - (l-1)\,\alpha\big]$ is an isomorphism of vector spaces since the pair $\boldsymbol{\Lambda}, l$ is separating. The fact that $e_{12}e_{21}$ is an isomorphism implies the lemma. □

2.1.5. **Theorem.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then for any $v_0 \in \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ there exist unique $v_1, \dots, v_l \in \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ such that the function*

$$v(x) \;=\; v_0\, x^l \,+\, v_1\, x^{l-1} \,+\, \dots \,+\, v_l$$

is a solution of the differential equation $\mathcal{D}_{\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}} v(x) \,=\, 0$.

Proof. If all weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$ are dominant integral, then the theorem holds by Theorem 12.1.3 from [MTV3]. By Lemma 2.1.4 the dimension of $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ does not depend on $\boldsymbol{\Lambda}$ if the pair $\boldsymbol{\Lambda}, l$ is separating. Hence the theorem holds for all separating $\boldsymbol{\Lambda}, l$. □

2.2. **Algebra A_D.**

2.2.1. Denote $\boldsymbol{a} \,=\, (a_1, \dots, a_l)$ and $\boldsymbol{h} \,=\, (h_1, \dots, h_n)$. Consider the space $\mathbb{C}^{l+n}$ with coordinates $\boldsymbol{a}, \boldsymbol{h}$. Denote by D the set of all points $\boldsymbol{p} \in \mathbb{C}^{l+n}$ whose coordinates satisfy the equations $q_{-1}(\boldsymbol{h}) = 0,\; q_0(\boldsymbol{h}) = 0$, where

$$q_{-1}(\boldsymbol{h}) \;=\; \sum_{s=1}^{n} h_s\,, \qquad q_0(\boldsymbol{h}) \;=\; \sum_{s=1}^{n} z_s h_s \,-\, l\,(\sum_{s=1}^{n} m_s + 1 - l)\;.$$

The set D is an affine space of dimension $l+n-2$.

2.2.2. Denote by $\mathcal{D}_{\boldsymbol{h}}$ the following polynomial differential operator in x depending on parameters $\boldsymbol{h}$,

$$\mathcal{D}_{\boldsymbol{h}} \;=\; \left(\prod_{s=1}^{n}(x-z_s)\right)\left(\frac{d^2}{dx^2}-\sum_{s=1}^{n}\frac{m_s}{x-z_s}\frac{d}{dx}+\sum_{s=1}^{n}\frac{h_s}{x-z_s}\right)\,. \tag{2.4}$$

If $\boldsymbol{p}\in D$, then the singular points of $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}$ are $z_1,\dots,z_n,\infty$ and the singular points are regular. For $s=1,\dots,n$, the exponents of $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}$ at z_s are $0, m_s+1$. The exponents of $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}$ at ∞ are $-l, l-1-\sum_{s=1}^n m_s$.

2.2.3. Denote by $p(x,\boldsymbol{a})$ the following polynomial in x depending on parameters $\boldsymbol{a}$,

$$p(x,\boldsymbol{a}) \;=\; x^l + a_1x^{l-1}+\dots+a_l\;.$$

If $\boldsymbol{h}$ satisfies equations $q_{-1}(\boldsymbol{h})=0$ and $q_0(\boldsymbol{h})=0$, then the polynomial $\mathcal{D}_{\boldsymbol{h}}(p(x,\boldsymbol{a}))$ is a polynomial in x of degree $l+n-3$,

$$\mathcal{D}_{\boldsymbol{h}}(p(x,\boldsymbol{a})) \;=\; q_1(\boldsymbol{a},\boldsymbol{h})\,x^{l+n-3} \;+\;\dots\;+\; q_{l+n-2}(\boldsymbol{a},\boldsymbol{h})\;.$$

The coefficients $q_i(\boldsymbol{a},\boldsymbol{h})$ are functions linear in $\boldsymbol{a}$ and linear in $\boldsymbol{h}$.

Denote by I_D the ideal in $\mathbb{C}[\boldsymbol{a},\boldsymbol{h}]$ generated by polynomials $q_{-1},q_0,q_1,\dots,q_{l+n-2}$. The ideal I_D defines a scheme $C_D\subset D$. Then

$$A_D \;=\; \mathbb{C}[\boldsymbol{a},\boldsymbol{h}]/I_D$$

is the algebra of functions on C_D.

The scheme C_D is the scheme of points $\boldsymbol{p}\in D$ such that the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}u(x)=0$ has a polynomial solution $p(x,\boldsymbol{a}(\boldsymbol{p}))$.

2.2.4. The scheme C_D and the algebra A_D depend on the choice of distinct numbers $\boldsymbol{z}=(z_1,\dots,z_n)$: $\quad C_D=C_D(\boldsymbol{z})$, $A_D=A_D(\boldsymbol{z})$.

2.2.5. **Theorem.** *Assume that the pair $\boldsymbol{\Lambda}$, l is separating. Then the dimension of $A_D(\boldsymbol{z})$, considered as a vector space, is finite and does not depend on the choice of distinct numbers $z_1,\dots,z_n$.*

Proof. It suffices to prove two facts:

(i) For any $\boldsymbol{z}$ with distinct coordinates there are no algebraic curves lying in $C_D(\boldsymbol{z})$.

(ii) Let a sequence $\boldsymbol{z}^{(i)}$, $i=1,2,\dots$, tend to a finite limit $\boldsymbol{z}=(z_1,\dots,z_n)$ with distinct $z_1,\dots,z_n$. Let $\boldsymbol{p}^{(i)}\in C_D(\boldsymbol{z}^{(i)})$, $i=1,2,\dots$, be a sequence of points. Then all coordinates $(\boldsymbol{a}(\boldsymbol{p}^{(i)}),\boldsymbol{h}(\boldsymbol{p}^{(i)}))$ remain bounded as i tends to infinity.

We prove (i), the proof of (ii) is similar.

For a point $\boldsymbol{p}$ in $C_D(\boldsymbol{z})$, the operator $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}$ has the form

$$B_0(x)\frac{d^2}{dx^2}+B_1(x)\frac{d}{dx}+B_2(x,\boldsymbol{p})$$

where the polynomials B_0, B_1, B_2 in x are of degree $n, n-1, n-2$, respectively, the top degree coefficients of the polynomials B_0, B_1, B_2 are equal to $1, -\sum_{s=1}^n m_s$, $l(\sum_{s=1}^n m_s + 1 - l)$, respectively, and the polynomials B_0, B_1 do not depend on $\boldsymbol{p}$.

Assume that (i) is not true. Then there exists a sequence of points $\boldsymbol{p}^{(i)} \in C_D(\boldsymbol{z})$, $i = 1, 2, \dots$, which tends to infinity as i tends to infinity.

Then it is easy to see that $\boldsymbol{h}(\boldsymbol{p}^{(i)})$ cannot tend to infinity since it would contradict to the fact that $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p}^{(i)})}(p(x, \boldsymbol{a}(\boldsymbol{p}^{(i)}))) = 0$.

Now choosing a subsequence we may assume that $\boldsymbol{h}(\boldsymbol{p}^{(i)})$ has finite limit as i tends to infinity.

If $\boldsymbol{h}(\boldsymbol{p}^{(i)})$ has finite limit as i tends to infinity, then $\boldsymbol{a}(\boldsymbol{p}^{(i)})$ cannot tend to infinity since it would mean that the limiting differential equation has a polynomial solution of degree less than l and this is impossible.

This reasoning implies that $\boldsymbol{p}^{(i)} \in C_D(\boldsymbol{z})$ cannot tend to infinity. Thus we get contradiction and statement (i) is proved. □

2.3. Second description of A_D.

2.3.1. **Theorem.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Assume that $\boldsymbol{h}$ satisfies equations $q_{-1}(\boldsymbol{h}) = 0$ and $q_0(\boldsymbol{h}) = 0$. Consider the system*

$$q_i(\boldsymbol{a}, \boldsymbol{h}) \ = \ 0 \ , \qquad i = 1, \dots, l \ , \tag{2.5}$$

as a system of linear equations with respect to $a_1, \dots, a_l$. Then this system has a unique solution $a_i = a_i(\boldsymbol{h})$, $i = 1, \dots, l$, where $a_i(\boldsymbol{h})$ are polynomials in $\boldsymbol{h}$. □

Proof. Theorem 2.3.1 follows from the fact that

$$q_i(\boldsymbol{a}, \boldsymbol{h}) \ = \ i\,(\sum_{s=1}^n m_s - 2l + i + 1)\, a_i \ + \ \sum_{j=1}^{i-1} q_{ij}(\boldsymbol{h})\, a_j$$

for $i = 1, \dots, l$. Here q_{ij} are some linear functions of $\boldsymbol{h}$. The coefficient of a_i does not vanish because the pair $\boldsymbol{\Lambda}, l$ is separating. □

2.3.2. Denote by I'_D the ideal in $\mathbb{C}[\boldsymbol{h}]$ generated by n polynomials q_{-1}, q_0, $q_j(\boldsymbol{a}(\boldsymbol{h}), \boldsymbol{h})$, $j = l+1, \dots, l+n-2$. Then

$$A_D \ \cong \ \mathbb{C}[\boldsymbol{h}]/I'_D \ .$$

2.4. Third description of A_D.

2.4.1. Assume that $h_1, \dots, h_n$ satisfy equations $q_{-1}(\boldsymbol{h}) = 0$, $q_0(\boldsymbol{h}) = 0$. Then

$$\sum_{s=1}^n \frac{h_s}{x - z_s} \ = \ \frac{g(x)}{(x - z_1) \dots (x - z_n)} \ ,$$

where

$$g(x) \;=\; l\,(\sum_{s=1}^{n} m_s + 1 - l)\,x^{n-2} + g_1(\boldsymbol{h})x^{n-3} + g_2(\boldsymbol{h})x^{n-2} + \dots + g_{n-2}(\boldsymbol{h})$$

for suitable $g_1(\boldsymbol{h}), \dots, g_{n-2}(\boldsymbol{h})$ which are linear functions in $\boldsymbol{h}$.

2.4.2. **Lemma.** *Let $c_1, \dots, c_{n-2}$ be arbitrary numbers. Consider the system of n linear equations*

$$\sum_{s=1}^{n} h_s \;=\; 0\,, \qquad \sum_{s=1}^{n} z_s h_s \;=\; l\,(\sum_{s=1}^{n} m_s + 1 - l)\ ,$$

$$g_i(\boldsymbol{h}) = c_i \qquad i = 1, \dots, n-2\ ,$$

with respect to $h_1, \dots, h_n$. Then this system has a unique solution. □

This lemma is the standard fact from the theory of simple fractions.

2.4.3. Let $\boldsymbol{g} = (g_0, \dots, g_{n-2})$ be a tuple of numbers and

$$g(x) \;=\; g_0 x^{n-2} + g_1 x^{n-3} + \dots + g_{n-2}\ .$$

The expression

$$(\prod_{s=1}^{n}(x - z_s))(\frac{d^2}{dx^2}p(x,\boldsymbol{a}) - \sum_{i=1}^{n} \frac{m_i}{x - z_i}\frac{d}{dx}p(x,\boldsymbol{a})) + g(x)p(x,\boldsymbol{a}) \;=\; 0\ .$$

is a polynomial in x of degree $l + n - 2$,

$$\hat{q}_0(\boldsymbol{a},\boldsymbol{g})\,x^{l+n-2} \;+\; \hat{q}_1(\boldsymbol{a},\boldsymbol{g})\,x^{l+n-3} \;+\; \dots \;+ \hat{q}_{l+n-2}(\boldsymbol{a},\boldsymbol{g})\ ,$$

where $\hat{q}_0(\boldsymbol{a},\boldsymbol{g}) \;=\; g_0 - l\,(\sum_{s=1}^{n} m_s + 1 - l)$.

2.4.4. **Lemma.** *The system of equations*

$$\hat{q}_i(\boldsymbol{a},\boldsymbol{g}) \;=\; 0\ , \qquad i = 0, \dots, n-2\ ,$$

determines $g_0, \dots, g_{n-2}$ uniquely as polynomials in $\boldsymbol{a}$. □

Proof. The equation $\hat{q}_0(\boldsymbol{a},\boldsymbol{g}) \;=\; 0$ gives $g_0 \;=\; l\,(\sum_{s=1}^{n} m_s + 1 - l)$. Now Lemma 2.4.4 follows from the fact that

$$\hat{q}_i(\boldsymbol{a},\boldsymbol{g}) \;=\; g_i \;+\; \sum_{j=1}^{i-1} \hat{q}_{ij}(\boldsymbol{a})g_j$$

for $i = 1, \dots, n-2$. Here $\hat{q}_{ij}$ are some linear functions of $\boldsymbol{a}$. □

2.4.5. Combining Lemmas 2.4.2 and 2.4.4, we obtain polynomial functions $h_i = h_i(\boldsymbol{a})$, $i = 1, \dots, n$.

Denote by I''_D the ideal in $\mathbb{C}[\boldsymbol{a}]$ generated by l polynomials $q_j(\boldsymbol{a}, \boldsymbol{h}(\boldsymbol{a}))$, $j = n-1, \dots, l+n-2$. Then

$$A_D \;\cong\; \mathbb{C}[\boldsymbol{a}]/I''_D\ .$$

2.5. **Epimorphism $\psi_{DM} : A_D \to A_M$.** Let $h_1, \dots, h_n$ be the functions on D, introduced in Section 2.2.1, and $H_1, \dots, H_n$ the Gaudin Hamiltonians.

2.5.1. **Theorem.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then the assignment $h_s \;\mapsto\; H_s$, $s = 1, \dots, n$, determines an algebra epimorphism $\psi_{DM} : A_D \to A_M$.*

Proof. The equations defining the scheme C_D are the equations of existence of a polynomial solution $p(x, \boldsymbol{a})$ of degree l to the polynomial differential equation $\mathcal{D}_{\boldsymbol{h}} u(x) = 0$. By Theorem 2.1.5, the defining equations for C_D are satisfied by the coefficients of the universal differential operator $\mathcal{D}_{\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}}$. □

3. Separation of variables

3.1. **Holomorphic representation.** The tensor product $M_{\boldsymbol{\Lambda}} = M_{\Lambda^{(1)}} \otimes \dots \otimes M_{\Lambda^{(n)}}$ of Verma $\mathfrak{gl}_2$-modules is identified with the space of polynomials $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ by the linear map

$$e_{21}^{j^1} v_{\Lambda^{(1)}} \otimes \dots \otimes e_{21}^{j^n} v_{\Lambda^{(n)}} \;\mapsto\; (x^{(1)})^{j^1} \dots (x^{(n)})^{j^n}\ ,$$

where $v_{\Lambda^{(s)}}$ is the generating vector of $M_{\Lambda^{(s)}}$. Then the $\mathfrak{gl}_2$-action on $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ is given by the differential operators,

$$e_{12}^{(s)} = -x^{(s)} \partial^2_{x^{(s)}} + m_s \partial_{x^{(s)}}\ , \qquad e_{21}^{(s)} = x^{(s)}\ ,$$

$$e_{11}^{(s)} = -2x^{(s)} \partial_{x^{(s)}} + m_s\ , \qquad e_{22}^{(s)} = 0\ ,$$

where $\partial_{x^{(s)}}$ denotes the derivative with respect to $x^{(s)}$.

3.2. **Change of variables.** Make the change of variables from $x^{(1)}, \dots, x^{(n)}$ to u, $y^{(1)}$, $\dots$, $y^{(n-1)}$ using the relation

$$\sum_{s=1}^{n} \frac{x^{(s)}}{t - z_s} \;=\; u\, \frac{\prod_{k=1}^{n-1} (t - y^{(k)})}{\prod_{s=1}^{n} (t - z_s)}\ ,$$

where t is an indeterminate. This relation defines $u, y^{(1)}, \dots, y^{(n-1)}$ uniquely up to permutation of $y^{(1)}, \dots, y^{(n-1)}$ unless $u = \sum_{s=1}^{n} x^{(s)} = 0$. The map $(u, y^{(1)}, \dots, y^{(n-1)}) \mapsto (x^{(1)}, \dots, x^{(n)})$ is an unramified covering on the complement to the union of diagonals $y^{(i)} = y^{(j)}$, $i \neq j$, and the hyperplane $u = 0$.

3.3. **Sklyanin's theorem.** Consider the operators $\widetilde{H}_1, \dots, \widetilde{H}_n$ defined by formula (2.2). Introduce the operators

$$K_i(\widetilde{H}) \;=\; \sum_{s=1}^{n} \frac{1}{y^{(i)} - z_s}\, \widetilde{H}_s \;, \qquad i = 1, \dots, n-1 \;.$$

3.3.1. **Theorem** [Sk]. *In variables $u, y^{(1)}, \dots, y^{(n-1)}$, we have*

$$K_i(\widetilde{H}) \;=\; -\,\partial^2_{y^{(i)}} \;+\; \sum_{s=1}^{n} \frac{m_s}{y^{(i)} - z_s}\,\partial_{y^{(i)}} \;, \qquad i = 1, \dots, n-1 \;.$$

3.4. **Universal weight function.** The weight subspace $M_{\boldsymbol\Lambda}[\Lambda^{(\infty)}] \subset M_{\boldsymbol\Lambda}$ is identified with the subspace of $\mathbb{C}[x^{(1)}, \dots, x^{(n)}]$ of homogeneous polynomials of degree l.

We consider the associated $M_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$-valued universal weight function

$$\prod_{j=1}^{l} \Big(\prod_{i=1}^{n} (t_j - z_i) \sum_{s=1}^{n} \frac{x^{(s)}}{t_j - z_s} \Big)$$

of variables $x^{(1)}, \dots, x^{(n)}$, $t_1, \dots, t_l$. In variables $u, y^{(1)}, \dots, y^{(n-1)}$, $t_1, \dots, t_l$, the universal weight function takes the form $(-1)^{ln}\, u^l \prod_{j=1}^{n-1} p(y^{(j)})$, where $p(x) = \prod_{i=1}^{l} (x - t_i)$. If we denote by $-a_1, a_2, \dots, (-1)^l a_l$ the elementary symmetric functions of $t_1, \dots, t_l$, then $p(x) = p(x, \boldsymbol{a})$ in notation of Section 2.2.3, and the universal weight function takes the form

$$\omega(u, \boldsymbol{y}, \boldsymbol{a}) \;=\; (-1)^{ln}\, u^l \prod_{j=1}^{n-1} p(y^{(j)}, \boldsymbol{a}) \;,$$

with $\boldsymbol{y} = (y^{(1)}, \dots, y^{(n-1)})$.

The trivial but important property of the universal weight function is given by the following lemma.

3.4.1. **Lemma.** *For every $\boldsymbol{p} \in D$, the vector $\omega(u, \boldsymbol{y}, \boldsymbol{a}(\boldsymbol{p}))$ is a nonzero vector of $M_{\boldsymbol\Lambda}[\Lambda^{(\infty)}]$.* □

Denote by ω_D the projection of the universal weight function $\omega(u, \boldsymbol{y}, \boldsymbol{a})$ to $M_{\boldsymbol\Lambda} \otimes A_D$.

3.4.2. **Theorem.** *For $s = 1, \dots, n$, we have*

$$\widetilde{H}_s\, \omega_D \;=\; h_s\, \omega_D \tag{3.1}$$

in $M_{\boldsymbol\Lambda} \otimes A_D$. *Moreover, we have*

$$\omega_D \;\in\; \operatorname{Sing} M_{\boldsymbol\Lambda}[\Lambda^{(\infty)}] \otimes A_D \;. \tag{3.2}$$

Proof. First we prove formula (3.1). Let $\mathbb{C}(u,\boldsymbol{y})$ be the algebra of rational functions in $u,\boldsymbol{y}$. For $i=1,\dots,n-1$, introduce

$$K_i(\boldsymbol{h}) \;=\; \sum_{s=1}^{n} \frac{h_s}{y^{(i)}-z_s} \;\in \mathbb{C}(u,\boldsymbol{y})\otimes A_D\;.$$

We claim that

$$K_i(\widetilde{H})\,\omega_D \;=\; K_i(\boldsymbol{h})\,\omega_D \tag{3.3}$$

in $\mathbb{C}(u,\boldsymbol{y})\otimes A_D$. Indeed,

$$K_i(\widetilde{H})\,\omega(u,\boldsymbol{y},\boldsymbol{a}) \;=\; (K_i(\boldsymbol{h})+K_i(\widetilde{H})-K_i(\boldsymbol{h}))\,\omega(u,\boldsymbol{y},\boldsymbol{a}) \;=\; K_i(\boldsymbol{h})\,\omega(u,\boldsymbol{y},\boldsymbol{a}) \;+$$
$$(-1)^{ln}\,u^l\left[\left(-\,\partial^2_{y^{(i)}} \;+\; \sum_{s=1}^{n}\frac{m_s}{y^{(i)}-z_s}\,\partial_{y^{(i)}} \;-\; \sum_{s=1}^{n}\frac{1}{y^{(i)}-z_s}\;h_s\right)p(y^{(i)},\boldsymbol{a})\right]\prod_{j\neq i}p(y^{(j)},\boldsymbol{a}).$$

Clearly, the last term has zero projection to $\mathbb{C}(u,\boldsymbol{y})\otimes A_D$ and we get formula (3.3).

Having formula (3.3), let us show that $\widetilde{H}_s\omega_D \;=\; h_s\omega_D$ in $\mathbb{C}[u,\boldsymbol{y}]\otimes A_D$. For that introduce two $\mathbb{C}[u,\boldsymbol{y}]\otimes A_D$-valued functions in a new variable x:

$$F_1(x) \;=\; \sum_{s=1}^{n}\frac{\widetilde{H}_s\omega_D}{x-z_s}\;, \qquad F_2(x) \;=\; \sum_{s=1}^{n}\frac{h_s\omega_D}{x-z_s}\;,$$

and show that the functions are equal.

Each of the functions is the ratio of a polynomial in x of degree $n-2$ and the polynomial $(x-z_1)\dots(x-z_n)$. To check that the two functions are equal it is enough to check that $F_1(x)=F_2(x)$ for $x=y^{(i)}$, $i=1,\dots,n-1$, but this follows from formula (3.3). Hence formula (3.1) is proved.

Formula (3.2) follows from formula (3.1). Indeed, by formula (2.2) we have $\sum_{s=1}^n z_s\widetilde{H}_s = \sum_{s=1}^n\sum_{r=1}^{s-1}(m_sm_r-\Omega_{s,r})$. This implies that $\sum_{s=1}^n z_s\widetilde{H}_s$ acts on the weight subspace $M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ as the operator $l(\sum_{s=1}^n m_s+1-l)-E_{21}E_{12}$, where $E_{ij}=\sum_{s=1}^n e^{(s)}_{ij}$. Since $\sum_{s=1}^n z_sh_s = l(\sum_{s=1}^n m_s+1-l)$, formula (3.1) allows us to conclude that $E_{21}E_{12}\,\omega_D=0$. The operator E_{21} is injective, in variables $u,y^{(1)},\dots,y^{(n-1)}$ it is the operator of multiplication by u. Therefore, $E_{12}\,\omega_D \;=\; 0$. □

4. Multiplication in A_D and Bethe algebra A_M

4.1. **Multiplication in A_D.** By Theorem 2.2.5, the scheme C_D considered as a set is finite, and the algebra A_D is the direct sum of local algebras corresponding to points $\boldsymbol{p}$ of the set C_D,

$$A_D \;=\; \oplus_{\boldsymbol{p}}\,A_{\boldsymbol{p},D}\;.$$

The local algebra $A_{\boldsymbol{p},D}$ may be defined as the quotient of the algebra of germs at $\boldsymbol{p}$ of holomorphic functions in $\boldsymbol{a},\boldsymbol{h}$ modulo the ideal $I_{\boldsymbol{p},D}$ generated by all functions $q_{-1},\dots,q_{l+n-2}$.

The local algebra $A_{\boldsymbol{p},D}$ contains the maximal ideal $\mathfrak{m}_{\boldsymbol{p}}$ generated by germs which are zero at $\boldsymbol{p}$.

For $f \in A_D$, denote by L_f the linear operator $A_D \to A_D$, $g \mapsto fg$, of multiplication by f. Consider the dual space

$$A_D^* \;=\; \oplus_{\boldsymbol{p}} A_{\boldsymbol{p},D}^*$$

and the dual operators $L_f^* : A_D^* \to A_D^*$. Every summand $A_{\boldsymbol{p},D}^*$ contains the distinguished one-dimensional subspace $\mathfrak{m}^{\boldsymbol{p}}$ which is the annihilator of $\mathfrak{m}_{\boldsymbol{p}}$.

4.1.1. **Lemma.**

(i) *For any point $\boldsymbol{p}$ of the scheme C_D considered as a set and any $f \in A_D$, we have $L_f^*(\mathfrak{m}^{\boldsymbol{p}}) \subset \mathfrak{m}^{\boldsymbol{p}}$.*

(ii) *For any point $\boldsymbol{p}$ of the scheme C_D considered as a set, if $W \subset A_{\boldsymbol{p},D}^*$ is a nonzero vector subspace invariant with respect to all operators L_f^*, $f \in A_D$, then W contains $\mathfrak{m}^{\boldsymbol{p}}$.*

Proof. For any $f \in \mathfrak{m}_{\boldsymbol{p}}$ we have $L_f^*(\mathfrak{m}^{\boldsymbol{p}}) = 0$. This gives part (i).

To prove part (ii) we consider the filtration of $A_{\boldsymbol{p},D}$ by powers of the maximal ideal,

$$A_{\boldsymbol{p},D} \supset \mathfrak{m}_{\boldsymbol{p}} \supset \mathfrak{m}_{\boldsymbol{p}}^2 \supset \cdots \supset \{0\} \ .$$

We consider a linear basis $\{f_{a,b}\}$ of $A_{\boldsymbol{p},D}$, $a = 0, 1, \dots$, $b = 1, 2, \dots$, which agrees with this filtration. Namely, we assume that for every i, the subset of all vectors $f_{a,b}$ with $a \geqslant i$ is a basis of $\mathfrak{m}_{\boldsymbol{p}}^i$.

Since dim $A_{\boldsymbol{p},T}/\mathfrak{m}_{\boldsymbol{p}} = 1$, there is only one basis vector with $a = 0$ and we also assume that this vector $f_{0,1}$ is the image of 1 in $A_{\boldsymbol{p},D}$.

Let $\{f^{a,b}\}$ denote the dual basis of $A_{\boldsymbol{p},D}^*$. Then the vector $f^{0,1}$ generates $\mathfrak{m}^{\boldsymbol{p}}$.

Let $w = \sum_{a,b} c_{a,b} f^{a,b}$ be a nonzero vector in W. Let a_0 be the maximum value of a such that there exists b with a nonzero $c_{a,b}$. Let b_0 be such that c_{a_0,b_0} is nonzero. Then it is easy to see that $L_{f_{a_0,b_0}}^* w = c_{a_0,b_0} f^{0,1}$. Hence W contains $\mathfrak{m}^{\boldsymbol{p}}$. $\square$

4.2. **Linear map $\tau : A_D^* \to$ Sing $M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.** Let $f_1, \dots, f_\mu$ be a basis of A_D considered as a vector space over $\mathbb{C}$. Write

$$\omega_D \;=\; \sum_i v_i \otimes f_i \qquad \text{with} \qquad v_i \in \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \ . \tag{4.1}$$

Denote by $V \subset \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ the vector subspace spanned by $v_1, \dots, v_\mu$. Define the linear map

$$\tau \;:\; A_D^* \;\to\; \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \ , \qquad g \;\mapsto\; g(\omega_D) = \sum_i \; g(f_i)\, v_i \ . \tag{4.2}$$

Clearly, V is the image of τ.

4.2.1. **Lemma.** *Let $\boldsymbol{p}$ be a point of C_D considered as a set. Let $\omega(u, \boldsymbol{y}, \boldsymbol{a}(\boldsymbol{p})) \in M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ be the value of the universal weight function at $\boldsymbol{p}$. Then the vector $\omega(u, \boldsymbol{y}, \boldsymbol{a}(\boldsymbol{p}))$ belongs to the image of τ.* □

4.2.2. **Lemma.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then for any $f \in A_D$ and $g \in A_D^*$, we have $\tau(L_f^*(g)) = \psi_{DM}(f)(\tau(g))$.*

In other words, the map τ intertwines the action of the algebra of multiplication operators L_f^* on A_D^* and the action on the Bethe algebra on $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.

Proof. The algebra A_D is generated by $h_1, \dots, h_n$. It is enough to prove that for any s we have $\tau(L_{h_s}^*(g)) = H_s(\tau(g))$. But $\tau(L_{h_s}^*(g)) = \sum_i g(h_s f_i) v_i = g(\sum_i v_i \otimes h_s f_i) = g(\sum_i H_s v_i \otimes f_i) = H_s(\tau(g))$. □

4.2.3. **Corollary.** *The vector subspace $V \subset \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ is invariant with respect to the action of the Bethe algebra A_M and the kernel of τ is a subspace of A_D^*, invariant with respect to multiplication operators L_f^*, $f \in A_D$.*

4.3. **First main theorem.**

4.3.1. **Theorem.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then the image of τ is* $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ *and the kernel of τ is zero.*

4.3.2. **Corollary.** *The map τ identifies the action of operators L_f^*, $f \in A_D$, on A_D^* and the action of the Bethe algebra on* $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. *Hence the epimorphism $\psi_{DM} : A_D \to A_M$ is an isomorphism.*

Proof of Theorem 4.3.1. Let $d = \dim \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Theorem 9.16 in [RV] says that for generic $\boldsymbol{z}$ there exists d points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d$ in C_D such that the vectors $\omega(u, \boldsymbol{y}, \boldsymbol{a}(\boldsymbol{p}_1)), \dots, \omega(u, \boldsymbol{y}, \boldsymbol{a}(\boldsymbol{p}_d))$ form a basis in $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Hence, τ is an epimorphism for generic $\boldsymbol{z}$ by Lemma 4.2.1. By Theorem 2.2.5 and Lemma 2.1.4 dimensions of A_D and $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ do not depend on $\boldsymbol{z}$. Hence $\dim A_D \geqslant \dim \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$. Therefore, to prove Theorem 4.3.1 it is enough to prove that τ has zero kernel.

Denote the kernel of τ by K. Let $A_D = \oplus_{\boldsymbol{p}} A_{\boldsymbol{p},D}$ be the decomposition into the direct sum of local algebras. Since K is invariant with respect to multiplication operators, we have $K = \oplus_{\boldsymbol{p}} K \cap A_{\boldsymbol{p},D}^*$ and for every $\boldsymbol{p}$ the vector subspace $K \cap A_{\boldsymbol{p},D}^*$ is invariant with respect to multiplication operators. By Lemma 4.1.1, if $K \cap A_{\boldsymbol{p},D}^*$ is nonzero, then $K \cap A_{\boldsymbol{p},D}^*$ contains the one-dimensional subspace $\mathfrak{m}^{\boldsymbol{p}}$.

Let $\{f_{a,b}\}$ be the basis of $A_{\boldsymbol{p},D}$ constructed in the proof of Lemma 4.1.1 and let $\{f^{a,b}\}$ be the dual basis of $A_{\boldsymbol{p},D}^*$. Then the vector $f^{0,1}$ generates $\mathfrak{m}^{\boldsymbol{p}}$. By definition of τ, the vector $\tau(f^{0,1})$ is equal to the value of the universal weight function at $\boldsymbol{p}$. By Lemma 3.4.1, this value is nonzero and that contradicts to the assumption that $f^{0,1} \in K$. □

4.4. **Grothendieck bilinear form on A_D.** Realize the algebra A_D as $\mathbb{C}[\boldsymbol{h}]/I'_D$, where I'_D is the ideal generated by n polynomials q_{-1}, q_0, $q_j(\boldsymbol{a}(\boldsymbol{h}), \boldsymbol{h})$, $j = l+1, \dots, l+n-2$, see Section 2.3.2.

Let $\rho : A_D \to \mathbb{C}$, be the Grothendieck residue,

$$f \ \mapsto \ \frac{1}{(2\pi i)^n} \operatorname{Res}_{C_D} \frac{f}{q_{-1}(\boldsymbol{h}) q_0(\boldsymbol{h}) \prod_{j=l+1}^{l+n-2} q_j(\boldsymbol{a}(\boldsymbol{h}), \boldsymbol{h})} \ .$$

Let $(\, , \,)_D$ be the Grothendieck symmetric bilinear form on A_D defined by the rule

$$(f, \, g)_D \ = \ \rho(fg) \ . \tag{4.3}$$

The Grothendieck bilinear form is non-degenerate.

The form $(\, , \,)_D$ determines a linear isomorphism $\phi : A_D \to A_D^*$, $f \mapsto (f, \, \cdot)_D$.

4.4.1. **Lemma.** *The isomorphism ϕ intertwines the operators L_f and L_f^* for any $f \in A_D$.*

Proof. For $g \in A_D$ we have $\phi(L_f(g)) = \phi(fg) = (fg, \cdot)_D = (g, f\cdot)_D = L_f^*((g, \cdot)_D) = L_f^* \phi(g)$. □

4.4.2. **Corollary.** *Assume that the pair $\boldsymbol{\Lambda}, l$ is separating. Then the composition $\tau\phi : A_D \to \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ is a linear isomorphism which intertwines the algebra of multiplication operators on A_D and the action of the Bethe algebra A_M on* $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.

5. Three more algebras

5.1. **New conditions on $\boldsymbol{\Lambda}$, l.** In the remainder of the paper we assume that $\boldsymbol{\Lambda} = (\Lambda^{(1)}, \dots, \Lambda^{(n)})$ is a collection of dominant integral $\mathfrak{gl}_2$-weights,

$$\Lambda^{(s)} \ = \ m_s \, \epsilon_1 \ , \qquad m_s \in \mathbb{Z}_{\geqslant 0} \ , \qquad s = 1, \dots, n \ . \tag{5.1}$$

We assume that $l \in \mathbb{Z}_{\geqslant 0}$ is such that the weight $\Lambda^{(\infty)} \ = \ \sum_{s=1}^n \Lambda^{(s)} \ - \ l\alpha$ is dominant integral. Hence the pair $\boldsymbol{\Lambda}, l$ is separating.

5.2. **Algebra A_P.** Denote $\tilde l = \sum_{s=1}^n m_s \ + 1 - l$. We have $\tilde l > l$. Denote

$$\tilde{\boldsymbol{a}} = (\tilde a_1, \dots, \tilde a_{\tilde l - l - 1}, \tilde a_{\tilde l - l + 1}, \dots, \tilde a_{\tilde l}) \ .$$

Consider space $\mathbb{C}^{\tilde l + l + n - 1}$ with coordinates $\tilde{\boldsymbol{a}}, \boldsymbol{a}, \boldsymbol{h}$, cf. Section 2.2.1.

Denote by $\tilde p(x, \tilde{\boldsymbol{a}})$ the following polynomial in x depending on parameters $\tilde{\boldsymbol{a}}$,

$$\tilde p(x, \tilde{\boldsymbol{a}}) \ = \ x^{\tilde l} + \tilde a_1 x^{\tilde l - 1} + \dots + \tilde a_{\tilde l - l - 1} x^{l+1} + \tilde a_{\tilde l - l + 1} x^{l-1} + \dots + \tilde a_{\tilde l} \ .$$

If $\boldsymbol{h}$ satisfies the equations $q_{-1}(\boldsymbol{h})=0$ and $q_0(\boldsymbol{h})=0$, then the polynomial $\mathcal{D}_{\boldsymbol{h}}(\tilde p(x,\tilde{\boldsymbol{a}}))$ is a polynomial in x of degree $\tilde l+n-3$,

$$\mathcal{D}_{\boldsymbol{h}}(\tilde p(x,\tilde{\boldsymbol{a}})) \ = \ \tilde q_1(\tilde{\boldsymbol{a}},\boldsymbol{h})\, x^{\tilde l+n-3} \ + \ \dots \ + \ \tilde q_{\tilde l+n-2}(\tilde{\boldsymbol{a}},\boldsymbol{h}) \ .$$

The coefficients $\tilde q_i(\tilde{\boldsymbol{a}},\boldsymbol{h})$ are functions linear in $\tilde{\boldsymbol{a}}$ and linear in $\boldsymbol{h}$.

Recall that if $p(x,\boldsymbol{a})=x^l+a_1x^{l-1}+\dots+a_l$ and $\boldsymbol{h}$ satisfies equations $q_{-1}(\boldsymbol{h})=0$ and $q_0(\boldsymbol{h})=0$, then the polynomial $\mathcal{D}_{\boldsymbol{h}}(p(x,\boldsymbol{a}))$ is a polynomial in x of degree $l+n-3$,

$$\mathcal{D}_{\boldsymbol{h}}(p(x,\boldsymbol{a})) \ = \ q_1(\boldsymbol{a},\boldsymbol{h})\, x^{l+n-3} \ + \ \dots \ + \ q_{l+n-2}(\boldsymbol{a},\boldsymbol{h}) \ .$$

Denote by I_P the ideal in $\mathbb{C}[\tilde{\boldsymbol{a}},\boldsymbol{a},\boldsymbol{h}]$ generated by polynomials $q_{-1},q_0,q_1,\dots,q_{l+n-2}$, $\tilde q_1,\dots,\tilde q_{\tilde l+n-2}$.

The ideal I_P defines a scheme $C_P\subset\mathbb{C}^{\tilde l+l+n-1}$. The algebra

$$A_P \ = \ \mathbb{C}[\tilde{\boldsymbol{a}},\boldsymbol{a},\boldsymbol{h}]/I_P$$

is the algebra of functions on C_P.

The scheme C_P is the scheme of points $\boldsymbol{p}\in\mathbb{C}^{\tilde l+l+n-1}$ such that the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}u(x)=0$ has two polynomial solutions $\tilde p(x,\tilde{\boldsymbol{a}}(\boldsymbol{p}))$ and $p(x,\boldsymbol{a}(\boldsymbol{p}))$.

5.3. **Algebra A_G.** Let d be a sufficiently large natural number and $\mathbb{C}_d[x]$ the vector subspace in $\mathbb{C}[x]$ of polynomials of degree not greater than d. Let G be the Grassmannian of all two-dimensional vector subspaces in $\mathbb{C}_d[x]$. Let $\boldsymbol{z}=(z_1,\dots,z_n)$ be distinct complex numbers.

For $s=1,\dots,n$, denote by $C_{z_s,\Lambda^{(s)}}\subset G$ the Schubert cycle associated with the point $z_s\in\mathbb{C}$ and weight $\Lambda^{(s)}$. The cycle $C_{z_s,\Lambda^{(s)}}$ is the closure of the set $C^o_{z_s,\Lambda^{(s)}}\subset G$ of all two-dimensional subspaces $V\subset\mathbb{C}_d[x]$ having a basis f_1,f_2 such that

$$f_1(z_s)=1 \qquad \text{and} \qquad f_2(x)=(x-z_s)^{m_s+1}+O((x-z_s)^{m_s+2}) \ .$$

Denote by $C_{\infty,\Lambda^{(\infty)}}\subset G$ the Schubert cycle associated with the point ∞ and weight $\Lambda^{(\infty)}$. $C_{\infty,\Lambda^{(\infty)}}$ is the closure of the set $C^o_{\infty,\Lambda^{(\infty)}}\subset G$ of all two-dimensional subspaces $V\subset\mathbb{C}_d[x]$ having a basis f_1,f_2 such that $\deg f_1=l$ and $\deg f_2=\tilde l$.

Consider the intersection

$$C_G \ = \ C_{\infty,\Lambda^{(\infty)}} \ \cap \ (\,\cap_{i=1}^n C_{z_i,\Lambda^{(i)}}\,) \ .$$

Denote by A_G the algebra of functions on C_G.

It is known from Schubert calculus that $\dim A_G$ is finite and does not depend on $\boldsymbol{z}$ with distinct coordinates.

5.3.1. It is easy to see that

$$C_G \ = \ C^o_{\infty,\Lambda^{(\infty)}} \ \cap \ (\,\cap_{i=1}^n C^o_{z_i,\Lambda^{(i)}}\,) \ .$$

5.3.2. We shall use the following presentation of the algebra A_G.

Consider space $\mathbb{C}^{\tilde{l}+l-1}$ with coordinates $\tilde{\boldsymbol{a}}, \boldsymbol{a}$. A point $\boldsymbol{p} \in \mathbb{C}^{\tilde{l}+l-1}$ will be called admissible if for every $s = 1, \dots, n$ at least one of the numbers $\tilde{p}(z_s, \tilde{\boldsymbol{a}}(\boldsymbol{p}))$, $p(z_s, \boldsymbol{a}(\boldsymbol{p}))$ is not zero. The set of all admissible points form a Zariski open subset $U \subset \mathbb{C}^{\tilde{l}+l-1}$.

For polynomials $f, g \in \mathbb{C}[x]$ denote by $\mathrm{Wr}(f,g)$ the Wronskian $f'g - fg'$, where $'$ denotes d/dx. The Wronskian of $\tilde{p}(x, \tilde{\boldsymbol{a}})$ and $p(x, \boldsymbol{a})$ has the form

$$\mathrm{Wr}\,(\tilde{p}(x, \tilde{\boldsymbol{a}}), p(x, \boldsymbol{a})) \;=\; (\tilde{l} - l)x^{\tilde{l}+l-1} + w_1(\tilde{\boldsymbol{a}}, \boldsymbol{a})x^{\tilde{l}+l-2} + \dots + w_{\tilde{l}+l-1}(\tilde{\boldsymbol{a}}, \boldsymbol{a})$$

for suitable polynomials $w_1, \dots, w_{\tilde{l}+l-1}$ in variables $\tilde{\boldsymbol{a}}, \boldsymbol{a}$.

Let us write

$$(\tilde{l} - l)\prod_{s=1}^{n}(x - z_s)^{m_s} \;=\; (\tilde{l} - l)x^{\tilde{l}+l-1} + c_1 x^{\tilde{l}+l-2} + \dots + c_{\tilde{l}+l-1}$$

for suitable numbers $c_1, \dots, c_{\tilde{l}+l-1}$.

Let A_U be the algebra of regular functions on the set U of all admissible points. Denote by $I_G \subset A_U$ the ideal generated by $\tilde{l} + l - 1$ polynomials $w_1 - c_1, \dots, w_{\tilde{l}+l-1} - c_{\tilde{l}+l-1}$. Then

$$A_G \;=\; A_U / I_G \;.$$

In this presentation of A_G the scheme C_G is the scheme of points $\boldsymbol{p} \in U$ such that the Wronskian of $\tilde{p}(x, \tilde{\boldsymbol{a}}(\boldsymbol{p}))$ and $p(x, \boldsymbol{a}(\boldsymbol{p}))$ is equal to $(\tilde{l} - l)\prod_{s=1}^{n}(x - z_s)^{m_s}$.

5.4. **Algebra A_L.** Let

$$L_{\boldsymbol{\Lambda}} \;=\; L_{\Lambda^{(1)}} \otimes \dots \otimes L_{\Lambda^{(n)}}$$

be the tensor product of irreducible $\mathfrak{gl}_2$-modules with highest weights $\Lambda^{(1)}, \dots, \Lambda^{(n)}$, respectively. Denote by $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ the subspace of $L_{\boldsymbol{\Lambda}}$ of singular vectors of weight $\Lambda^{(\infty)}$.

Let S denote the tensor Shapovalov form on $\mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$, induced from the tensor product of the Shapovalov forms on the factors of $M_{\boldsymbol{\Lambda}} = M_{\Lambda^{(1)}} \otimes \dots \otimes M_{\Lambda^{(n)}}$.

The Shapovalov form determines the linear epimorphism

$$\sigma \;:\; \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \;\to\; \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \;.$$

The Bethe algebra A_M preserves the kernel of σ and induces a commutative subalgebra A_L in $\mathrm{End}\,(\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}])$ called the Bethe algebra on $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.

Denote by $\psi_{ML} : A_M \to A_L$ the corresponding epimorphism.

5.4.1. Denote by

$$\mathcal{D}_L \;=\; \frac{d^2}{dx^2} \;-\; \sum_{s=1}^{n} \frac{m_s}{x - z_s}\frac{d}{dx} \;+\; \sum_{s=1}^{n} \frac{\psi_{ML}(H_s)}{x - z_s}$$

the universal differential operator associated with the subspace $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and collection $\boldsymbol{z}$.

5.4.2. **Theorem.** *Assume that the pair* $\boldsymbol{\Lambda}, l$ *satisfies conditions of Section 5.1. Then for any* $v_0 \in \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ *there exist* $v_1, \dots, v_{\tilde l} \in \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ *such that the function*

$$v(x) \;=\; v_0\, x^{\tilde l} \,+\, v_1\, x^{\tilde l -1} \,+\, \dots \,+\, v_{\tilde l}$$

is a solution of the differential equation $\mathcal{D}_L v(x) \,=\, 0$.

This theorem is a particular case of Theorem 12.3 in [MTV3].

6. Four more homomorphisms

6.1. **Isomorphism** $\psi_{GP} : A_G \to A_P$**.** A point $\boldsymbol{p}$ of C_P defines the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})} u(x) = 0$ and two solutions $\tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{p}))$ and $p(x, \boldsymbol{a}(\boldsymbol{p}))$. We have

$$\operatorname{Wr}\left(\tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{p})), p(x, \boldsymbol{a}(\boldsymbol{p}))\right) \;=\; (\tilde l - l) \prod_{s=1}^{n} (x - z_s)^{m_s} \ .$$

Hence, the pair $\tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{p}))$, $p(x, \boldsymbol{a}(\boldsymbol{p}))$ defines a point of C_G.

This construction defines a homomorphism of algebras $\psi_{GP} : A_G \to A_P$.

6.1.1. **Theorem.** *The homomorphism* ψ_{GP} *is an isomorphism.*

Proof. We construct the inverse homomorphism as follows. Let $\boldsymbol{v}$ be a point of C_G. Consider the following differential equation with respect to a function $u(x)$,

$$\det \begin{pmatrix} u'' & u' & u \\ \tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{v}))'' & \tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{v}))' & \tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{v})) \\ p(x, \boldsymbol{a}(\boldsymbol{v}))'' & p(x, \boldsymbol{a}(\boldsymbol{v}))' & p(x, \boldsymbol{a}(\boldsymbol{v})) \end{pmatrix} \;=\; 0 \ .$$

Let us write this differential equation as $B_0(x) u'' + B_1(x) u' + B_2(x) u = 0$. Here

$$B_0(x) \;=\; \operatorname{Wr}(\tilde p(x, \tilde{\boldsymbol{a}}(\boldsymbol{v})), p(x, \boldsymbol{a}(\boldsymbol{v}))) \;=\; (\tilde l - l) \prod_{s=1}^{n} (x - z_s)^{m_s} \ .$$

It is easy to see that each of the polynomials B_1, B_2 is divisible by the polynomial

$$B(x) \;=\; (\tilde l - l) \prod_{s=1}^{n} (x - z_s)^{m_s - 1} \ .$$

Introduce the differential operator

$$\mathcal{D}_{\boldsymbol{v}} \;=\; b_0(x) \frac{d^2}{dx^2} + b_1(x) \frac{d}{dx} + b_2(x) \;=\; \frac{1}{B(x)} \left(B_0(x) \frac{d^2}{dx^2} + B_1(x) \frac{d}{dx} + B_2(x) \right) \ .$$

Then

$$b_0(x) \;=\; \prod_{s=1}^{n} (x - z_s) \ , \qquad b_1(x) \;=\; \prod_{s=1}^{n} (x - z_s) \left(\sum_{s=1}^{n} \frac{-m_s}{x - z_s} \right) \ ,$$

and $b_2(x)$ is a polynomial of degree $n-2$, whose leading coefficient is $\tilde l l$.

The triple, consisting of the differential operator $\mathcal{D}_{\boldsymbol{v}}$ and two polynomials $\tilde{p}(x, \tilde{\boldsymbol{a}}(\boldsymbol{v}))$ and $p(x, \boldsymbol{a}(\boldsymbol{v}))$, determines a point of C_P, thus defining the inverse homomorphism $A_P \to A_G$. $\square$

6.1.2. **Corollary.** *The dimension of the algebra A_P is finite and does not depend on $\boldsymbol{z}$ with distinct coordinates.*

Indeed, dim $A_P =$ dim A_G and dim A_G is finite and does not depend on $\boldsymbol{z}$ with distinct coordinates.

6.1.3. It is known from Schubert calculus that dim $A_G =$ dim Sing $L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.

6.2. **Epimorphism $\psi_{DP} : A_D \to A_P$.** A point $\boldsymbol{p}$ of C_P determines the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}\, u(x) \,=\, 0$ and two solutions $\tilde{p}(x, \tilde{\boldsymbol{a}}(\boldsymbol{p}))$ and $p(x, \boldsymbol{a}(\boldsymbol{p}))$. Then the pair, consisting of the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}\, u(x) \,=\, 0$ and one of the solutions $p(x, \boldsymbol{a}(\boldsymbol{p}))$ determines a point of C_D. This correspondence defines a natural algebra epimorphism $\psi_{DP} : A_D \to A_P$.

6.3. **Linear map $\xi : A_D \to \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$.** Denote by $\xi : A_D \to \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ the composition of linear maps

$$A_D \xrightarrow{\phi} A_D^* \xrightarrow{\tau} \mathrm{Sing}\, M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \xrightarrow{\sigma} \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]\ .$$

By Theorem 4.3.1, ξ is a linear epimorphism.

Denote by $\psi_{DL} : A_D \to A_L$ the algebra epimorphism defined as the composition $\psi_{ML}\psi_{DM}$.

6.3.1. **Lemma.** *The linear map ξ intertwines the action of the multiplication operators L_f, $f \in A_D$, on A_D and the action of the Bethe algebra A_L on* Sing $L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$*, i.e. for any $f, g \in A_D$ we have $\xi(L_f(g)) \,=\, \psi_{DL}(f)(\xi(g))$.*

The lemma follows from Corollary 4.4.2.

6.3.2. **Lemma.** *The kernel of ξ coincides with the kernel of ψ_{DL}.*

Proof. If $\psi_{DL}(f) = 0$, then $\xi(f) = \xi(L_f(1)) = \psi_{DL}(f)(\xi(1)) = 0$. On the other hand, if $\xi(f) = 0$, then for any $g \in A_D$ we have $\psi_{DL}(f)(\xi(g)) = \xi(L_f(g)) = \xi(fg) = \xi(L_g(f)) = \psi_{DL}(g)(\xi(f)) = 0$. Since ξ is an epimorphism, this means that $\psi_{DL}(f) = 0$. $\square$

6.3.3. **Lemma.** *The kernel of ξ coincides with the kernel of ψ_{DP}.*

Proof. By Schubert calculus dim Sing $L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}] \,=\,$ dim A_G. Hence it suffices to show that the kernel of ξ contains the kernel of ψ_{DP}. But this follows from Theorems 2.1.5 and 5.4.2.

Indeed the defining relations in $A_P = A_D/(\ker\, \psi_{DP})$ are the conditions on the operator $\mathcal{D}_{\boldsymbol{h}}$ to have two linearly independent polynomials in the kernel. Theorems 2.1.5 and 5.4.2 guarantee these relations for elements of the Bethe algebra A_L. Hence, the kernel of ψ_{DL} contains the kernel of ψ_{DP}. By Lemma 6.3.2, the kernel of ξ coincides with the kernel of ψ_{DL}. Therefore, the kernel of ξ contains the kernel of ψ_{DP}. $\square$

6.3.4. **Corollary.** *Since the algebra epimorphisms ψ_{DP} and ψ_{DL} have the same kernels, the algebras A_P and A_L are isomorphic, and hence by Theorem 6.1.1 the algebras A_G and A_L are isomorphic.* □

6.4. **Second main theorem.** Denote by $\psi_{PL} : A_P \to A_L$ the isomorphism induced by ψ_{DL} and ψ_{DP}. The previous lemmas imply the following theorem.

6.4.1. **Theorem.** *The linear map ξ induces a linear isomorphism*

$$\zeta \ : \ A_P \ \to \ \mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$$

which intertwines the multiplication operators L_f, $f \in A_P$, on A_P and the action of the Bethe algebra A_L on $\mathrm{Sing}\, L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$, i.e. for any $f, g \in A_P$ we have $\zeta(L_f(g)) = \psi_{PL}(f)(\zeta(g))$. □

6.4.2. **Corollary.** *If every operator $f \in A_L$ is diagonalizable, then the algebra A_L has simple spectrum and all of the points of the intersection of Schubert cycles*

$$C_G \ = \ C_{\infty,\Lambda^{(\infty)}} \ \cap \ (\cap_{i=1}^n C_{z_i,\Lambda^{(i)}})$$

are of multiplicity one.

Proof of Corollary. The algebras A_L, A_P and A_G are all isomorphic. We have $A_P = \oplus_{\boldsymbol{p}} A_{\boldsymbol{p},P}$ where the sum is over the points of the scheme C_P considered as a set and $A_{\boldsymbol{p},P}$ is the local algebra associated with a point $\boldsymbol{p}$. The algebra $A_{\boldsymbol{p},P}$ has nonzero nilpotent elements if $\dim A_{\boldsymbol{p},P} > 1$. If every element $f \in A_P$ is diagonalizable, then the algebra A_P is the direct sum of one-dimensional local algebras. Hence A_P has simple spectrum as well as the algebras A_L and A_G. □

6.4.3. Corollary 6.4.2 has the following application.

Corollary [EGSV]. *If $z_1, \dots, z_n$ are real and distinct, then all of the points of the intersection of Schubert cycles*

$$C_G \ = \ C_{\infty,\Lambda^{(\infty)}} \ \cap \ (\cap_{i=1}^n C_{z_i,\Lambda^{(i)}})$$

are of multiplicity one.

Proof. If $z_1, \dots, z_n$ are real and distinct, then by Corollary 3.5 in [MTV2] all elements of the Bethe algebra A_L are diagonalizable operators. Hence the spectrum of A_G is simple and all points of C_G are of multiplicity one. □

This corollary is proved in [EGSV] by a different method.

7. Operators with polynomial kernel and Bethe algebra A_L

7.1. **Linear isomorphism $\theta : A_P^* \to \mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$.** Define the symmetric bilinear form on A_P by the formula

$$(f,\, g)_P \;=\; S(\zeta(f),\, \zeta(g)) \qquad \text{for all} \quad f, g \in A_P\ .$$

Recall that $S(\,,\,)$ denotes the Shapovalov form.

7.1.1. **Lemma.** *The form $(\,,\,)_P$ is non-degenerate.*

The lemma follows from the fact that the Shapovalov form on $\mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$ is non-degenerate and the fact that ζ is an isomorphism.

7.1.2. **Lemma.** *We have $(fg, h)_P = (g, fh)_P$ for all $f, g, h \in A_P$.* □

The form $(\,,\,)_P$ defines a linear isomorphism $\pi : A_P \to A_P^*$, $f \mapsto (f\,, \cdot)_P$.

7.1.3. **Corollary.** *The map π intertwines the multiplication operators L_f, $f \in A_P$, on A_P and the dual operators L_f^*, $f \in A_P$, on A_P^*.*

7.2. **Third main theorem.** Summarizing Theorem 6.4.1 and Corollary 7.1.3 we obtain the following theorem.

7.2.1. **Theorem.** *The composition $\theta = \zeta\pi^{-1}$ is a linear isomorphism from A_P^* to $\mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$ which intertwines the multiplication operators L_f^*, $f \in A_P$, on A_P^* and the action of the Bethe algebra A_L on $\mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$, i.e. for any $f \in A_P$ and $g \in A_P^*$ we have $\theta(L_f^*(g)) \,=\, \psi_{PL}(f)(\theta(g))$.* □

7.2.2. Assume that $v \in \mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$ is an eigenvector of the Bethe algebra A_L, that is, $\psi_{ML}(H_s)v = \lambda_s v$ for suitable $\lambda_s \in \mathbb{C}$ and $s = 1, \dots, n$. Then, by Corollaries 12.2.1 and 12.2.2 in [MTV3], the differential operator

$$\mathcal{D} \;=\; \frac{d^2}{dx^2} \;-\; \sum_{s=1}^{n} \frac{m_s}{x - z_s}\,\frac{d}{dx} \;+\; \sum_{s=1}^{n} \frac{\lambda_s}{x - z_s}$$

has the following properties. The operator $\mathcal{D}$ has regular singular points at $z_1, \dots, z_n, \infty$. For $s = 1, \dots, n$, the exponents of $\mathcal{D}$ at z_s are $0, m_s + 1$. The exponents of $\mathcal{D}$ at ∞ are $-l, l-1-\sum_{s=1}^{n} m_s$. The kernel of $\mathcal{D}$ consists of polynomials only. The following corollary of Theorem 7.2.1 gives the converse statement.

7.2.3. **Corollary of Theorem 7.2.1.** *Let $\boldsymbol{p} \in \mathbb{C}^n$ be a point such that $q_{-1}(\boldsymbol{h}(\boldsymbol{p})) = 0$, $q_0(\boldsymbol{h}(\boldsymbol{p})) = 0$, and all solutions of the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})}u(x) = 0$ are polynomials. Then there exists an eigenvector $v \in \mathrm{Sing}\, L_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$ of the action of the Bethe algebra A_L such that for every $s = 1, \dots, n$ we have*

$$\psi_{ML}(H_s)\, v \;=\; h_s(\boldsymbol{p})\, v\ .$$

Proof of Corollary 7.2.3. Indeed, such $\boldsymbol{p}$ defines a linear function $\eta : A_P \to \mathbb{C}$, $h_s \mapsto h_s(\boldsymbol{p})$ for $s = 1, \dots, n$. Moreover, $\eta(fg) = \eta(f)\eta(g)$ for all $f, g \in A_P$. Hence $\eta \in A_P^*$ is an eigenvector of multiplication operators on A_P^*. By Theorem 7.2.1 this eigenvector corresponds to an eigenvector $v \in \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ of the action of the Bethe algebra A_L with eigenvalues prescribed in Corollary 7.2.3. □

7.2.4. Assume that $\boldsymbol{p} \in \mathbb{C}^n$ is a point satisfying the assumptions of Corollary 7.2.3. We describe how to find the eigenvector $v \in \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ indicated in Corollary 7.2.3.

Let $f(x)$ be the monic polynomial of degree l which is a solution of the differential equation $\mathcal{D}_{\boldsymbol{h}(\boldsymbol{p})} w(x) = 0$. Consider the polynomial

$$\omega(u, \boldsymbol{y}) \;=\; u^l \prod_{j=1}^{n-1} f(y^{(j)})$$

as an element of $M_{\boldsymbol{\Lambda}}$, see Section 3.4. By Theorem 3.4.2 this vector lies in $\operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ and $\omega(u, \boldsymbol{y})$ is an eigenvector of the Bethe algebra A_M with eigenvalues presecribed in Corollary 7.2.3. Consider the maximal subspace $V \subset \operatorname{Sing} M_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ with three properties: i) V contains $\omega(u, \boldsymbol{y})$, ii) V does not contain other eigenvectors of the Bethe algebra A_M, iii) V is invariant with respect to the Bethe algebra A_M. Let $\sigma(V) \subset \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ be the image of V under the epimorphism σ. Then the subspace $\sigma(V)$ contains a unique one-dimensional subspace of eigenvectors of the Bethe algebra A_L. Any such an eigenvector may serve as an eigenvector of the Bethe algebra A_L indicated in Corollary 7.2.3.

8. Appendix. Grothendieck and Shapovalov forms

8.1. **Form $(\,,\,)_S$ on A_D.** Define the symmetric bilinear form on A_D by the formula

$$(f,\, g)_S \;=\; S(\xi(f),\, \xi(g)) \qquad \text{for all} \quad f, g \in A_D\,,$$

where $S(\,,\,)$ denotes the Shapovalov form.

8.1.1. **Lemma.** *The kernel of the bilinear form $(\,,\,)_S$ coincides with the kernel of the linear map ξ.*

The lemma follows from the fact that the Shapovalov form on $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\Lambda^{(\infty)}]$ is non-degenerate.

8.1.2. **Lemma.** *We have $(fg, h)_S = (g, fh)_S$ for all $f, g, h \in A_D$.*

The lemma follows from Theorem 4.3.1 and the fact that the operators of the Bethe algebra are symmetric with respect to the Shapovalov form, see, for example, [RV] and [MTV1].

8.1.3. **Corollary.** *There exists $F \in A_D$ such that $(f, g)_S \;=\; (Ff, g)_D$ for all $f, g \in A_F$.*

8.1.4. **Lemma.** *The kernel of the multiplication operator $L_F : A_D \to A_D$ coincides with the kernel of ξ.*

The lemma follows from Theorem 4.3.1 and the fact that the kernel of σ is the kernel of the Shapovalov form on $\mathrm{Sing}\, M_{\mathbf{\Lambda}}[\Lambda^{(\infty)}]$.

The image of L_F is the principal ideal $(F) \subset A_D$ generated by F.

8.1.5. **Corollary.** *The algebra of operators $L_f, f \in A_D$, restricted to (F) is isomorphic to the algebra A_L.*

Denote $J = \{f \in A_D \mid fg = 0 \text{ for all } g \in \ker \psi_{DP}\}$. The following lemma describes the ideal (F) without using the Shapovalov form.

8.1.6. **Lemma.** *We have $(F) = J$.*

Proof. The inclusion $(F) \subset J$ follows from Lemmas 8.1.4 and 6.3.3. On the other hand, since $(\ ,\ )_D$ is non-degenerate, we have $\dim J = \dim A_D - \dim \ker \psi_{DP}$. By Lemma 8.1.4, (F) has the same dimension and hence $(F) = J$. $\square$

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